Available online: February 26, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 125–135 (2019)
DOI: 10.31801/cfsuasmas.443665
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

# BOUNDS FOR INITIAL MACLAURIN COEFFICIENTS OF A SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SUBORDINATION

AHMAD MOTAMEDNEZHAD, SHAHPOUR NOSRATI, AND SIMA ZAKER

Abstract. In this paper, we investigate the bounds of the coefficients for new subclasses of analytic and bi-univalent functions in the open unit disc defined by subordination. The coefficients bounds presented in this paper would generalize and improve those in related works of several earlier authors

## 1. Introduction and definitions

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

The Koebe one-quarter theorem [8] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

Received by the editors: October 05, 2017, Accepted: November 20, 2017.
2010 *Mathematics Subject Classification.* 30C45; 30C50.
*Key words and phrases.* Analytic functions, bi-univalent functions, boefficient estimates, Hadamard product, subordination.

Recently some researchers have been devoted to study the bi-univalent functions class $\Sigma$ and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class $\Sigma$, see [15]. In fact that this widely-cited work by Srivastava et al. [15] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al.[6, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], and others [10, 11, 14, 27]. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \Sigma$, is still an open problem. In fact there is no direct way to get bound for coefficients greater than three. In special cases there are some papers in which the Faber polynomial methods were used for determining upper bounds for higher-order coefficients ( for example see [20]).

More recently El-Ashwah [10] introduced the following two subclasses of the bi-univalent function class $\Sigma$ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

**Definition 1.** [10] *For $0 < \alpha \leq 1;\ \lambda \geq 1$, a function $f(z)$ given by* (1) *is said to be in the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma,\ |arg((1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z))| < \frac{\alpha\pi}{2} \quad (z \in \mathcal{U}), \tag{3}$$

*and*

$$|arg((1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w))| < \frac{\alpha\pi}{2} \quad (w \in \mathcal{U}), \tag{4}$$

where the functions $h(z)$ and $(f*h)^{-1}(w)$ are defined by:

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n \qquad (h_n > 0), \tag{5}$$

and

$$\begin{aligned}(f*h)^{-1}(w) =& w - a_2h_2w^2 + (2a_2^2h_2^2 - a_3h_3)w^3 \\ &- (5a_2^3h_2^3 - 5a_2h_2a_3h_3 + a_4h_4)w^4 + \cdots .\end{aligned} \tag{6}$$

**theorem 1.** [10] *Let $f(z)$ given by* (1) *be in the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{h_2\sqrt{(\lambda+1)^2 + \alpha(1 + 2\lambda - \lambda^2)}},\ |a_3| \leq \frac{1}{h_3}\left(\frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{(2\lambda+1)}\right). \tag{7}$$

**Definition 2.** [10] *For $0 \leq \beta < 1$; $\lambda \geq 1$, a function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(h, \beta, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma, \; Re\left((1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z)\right) > \beta \; ( \; z \in \mathcal{U}), \tag{8}$$

*and*

$$Re\left((1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w)\right) > \beta \; ( \; w \in \mathcal{U}), \tag{9}$$

*where the functions $h(z)$ and $(f*h)^{-1}(w)$ are defined by (5) and (6) respectively.*

**theorem 2.** [10] *Let $f(z)$ given by (1) be in the class $\mathcal{B}_\Sigma(h, \beta, \lambda)$, $0 \leq \beta < 1$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(2\lambda+1)}}, \; |a_3| \leq \frac{1}{h_3}\left(\frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{(2\lambda+1)}\right). \tag{10}$$

The object of the present paper is to introduce a new subclasses of the function class $\Sigma$ and obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass. Our results would generalize and improve the Theorem 1 and Theorem 2.

## 2. Subclass $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$

An analytic function $f$ is said to be subordinate to another analytic function $g$, written as

$$f(z) \prec g(z) \quad (z \in \mathcal{U}),$$

if there exists a Schwarz function $w$, which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathcal{U})$, such that $f(z) = g(w(z))$. In particular, if the function $g$ is univalent in $\mathcal{U}$, then we have the following equivalence:
$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $\varphi$ be an analytic function with positive real part in $\mathcal{U}$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathcal{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{11}$$

We now introduce the following class of bi-univalent functions.

**Definition 3.** Let $0 \leq \gamma \leq 1$ and $\tau \in \mathbb{C} - \{0\}$. A function $f \in \Sigma$ given by (1), is said to be in the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ if each of the following subordinate conditions holds true:

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) - 1\right] \prec \varphi(z) \quad (z \in \mathcal{U}), \tag{12}$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)^{-1}(w)}{w}+\lambda((f*h)^{-1})'(w)-1\right]\prec\varphi(w)\quad(w\in\mathcal{U}),\tag{13}$$

where the functions $h(z)$ and $(f*h)^{-1}(w)$ are defined by (5) and (6) respectively.

**Remark 1.** *There are many choices of $\varphi$ which would provide interesting subclasses of class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$.*

- *If we take $\tau=1$ and $\varphi=\left(\frac{1+z}{1-z}\right)^\alpha$ $(0<\alpha\le 1)$, then the class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$ reduce to Definition 1.*
- *If we take $\tau=1$, $\varphi=\left(\frac{1+z}{1-z}\right)^\alpha$ $(0<\alpha\le 1)$ and $h(z)=\frac{z}{1-z}$, then the class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$ reduce to the class $\mathcal{B}_\Sigma(\alpha,\lambda)$ introduced and studied by Frasin and Aouf* [11].
- *If we take $\tau=\lambda=1$, $\varphi=\left(\frac{1+z}{1-z}\right)^\alpha$ $(0<\alpha\le 1)$ and $h(z)=\frac{z}{1-z}$, then the class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$ reduce to the class $\mathcal{H}_\Sigma^\alpha$ introduced and studied by Srivastava et al.* [15].
- *If we take $\tau=1$ and $\varphi=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, then the class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$ reduce to Definition 2.*
- *If we take $\tau=1$ and $\varphi=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, then the class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$ reduce to the class $\mathcal{B}_\Sigma(\beta,\lambda)$ introduced and studied by Frasin and Aouf* [11].
- *If we take $\tau=\lambda=1$ and $\varphi=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, then the class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$ reduce to the class $\mathcal{H}_\Sigma(\beta)$ introduced and studied by Srivastava et al.* [15].

## 3. Coefficient bounds for the class $\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$

In order to derive our main results, we have to recall here the following lemma.

**Lemma 1.** [13] *Let $p\in\mathcal{P}$ the family of all functions $p$ analytic in $\mathcal{U}$ for which $\Re p(z)>0$ and have the form $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots\quad(z\in\mathcal{U})$ for $z\in\mathcal{U}$. Then $|p_n|\le 2$, for each $n$.*

**theorem 3.** *Let $f(z)\in\mathcal{B}_\Sigma(\varphi,\tau,\lambda)$ be of the form* (1)*. Then*

$$|a_2|\le\min\left[\frac{|\tau|B_1}{h_2(1+\lambda)},\frac{1}{h_2}\sqrt{\frac{|\tau|(B_1+|B_1-B_2|)}{(1+2\lambda)}},\frac{|\tau|B_1\sqrt{B_1}}{h_2\sqrt{|(B_1-B_2)(1+\lambda)^2+\tau B_1^2(1+2\lambda)|}}\right].\tag{14}$$

$$|a_3|\le\min\left[\frac{|\tau|(B_1+|B_1-B_2|)}{h_3(1+2\lambda)},\frac{|\tau^2|B_1^2}{h_3(1+\lambda)^2}+\frac{|\tau|B_1}{h_3(1+2\lambda)}\right].\tag{15}$$

*where the coefficients $B_1$ and $B_2$ are given as in* (11)*.*

*Proof.* For $f \in \mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ , there are analytic functions $u, v : \mathcal{U} \longrightarrow \mathcal{U}$, with $u(0) = v(0) = 0$, satisfying the following conditions:

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) - 1\right] = \varphi(u(z)) \quad (z \in \mathcal{U}) \tag{16}$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w) - 1\right] = \varphi(v(w)) \quad (w \in \mathcal{U}). \tag{17}$$

Now we define the functions $p_1$ and $p_2$ by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{18}$$

and

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots . \tag{19}$$

Then $p_1$ and $p_2$ are analytic in $\mathcal{U}$ with positive real parts and $p_1(0) = 1 = p_2(0)$. Therefore, in view of the Lemma 1, we have

$$|b_n| \leq 2 \ \ \textit{and} \ \ |c_n| \leq 2 \quad (n \in \mathbb{N}). \tag{20}$$

Solving (18) and (19) for $u(z)$ and $v(z)$, we get

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right] \quad (z \in \mathbb{U}) \tag{21}$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{b_1^2}{2}\right)z^2 + \cdots\right] \quad (z \in \mathbb{U}). \tag{22}$$

Clearly, upon substituting from (21) and (22) into (16) and (17), respectively, if we make use of (11), we find that

$$\begin{aligned} &1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) - 1\right] = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \\ &= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \cdots \end{aligned} \tag{23}$$

and

$$\begin{aligned} &1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w) - 1\right] = \varphi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) \\ &= 1 + \frac{1}{2}B_1 b_1 w + \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right]w^2 + \cdots . \end{aligned} \tag{24}$$

Clearly it follow from (23) and (24), we have

$$\frac{(1+\lambda)a_2h_2}{\tau} = \frac{1}{2}B_1c_1, \tag{25}$$

$$\frac{(1+2\lambda)a_3h_3}{\tau} = \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right], \tag{26}$$

$$\frac{-(1+\lambda)a_2h_2}{\tau} = \frac{1}{2}B_1b_1 \tag{27}$$

and

$$\frac{(1+2\lambda)(2a_2^2h_2^2 - a_3h_3)}{\tau} = \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2\right]. \tag{28}$$

From (25) and (27), it follows that

$$c_1 = -b_1 \tag{29}$$

and

$$\frac{2(1+\lambda)^2a_2^2h_2^2}{\tau^2} = \frac{1}{4}B_1^2(c_1^2 + b_1^2). \tag{30}$$

By using lemma 1, we obtain

$$|a_2| \leq \frac{|\tau|B_1}{h_2(1+\lambda)}. \tag{31}$$

By adding (26) and (28) we have

$$\frac{(1+2\lambda)2a_2^2h_2^2}{\tau} = \left[\frac{1}{2}B_1(b_2+c_2) - \frac{1}{4}B_1(b_1^2+c_1^2) + \frac{1}{4}B_2(b_1^2+c_1^2)\right]. \tag{32}$$

i.e.,

$$a_2^2 = \frac{\tau\left[2B_1(b_2+c_2) - B_1(b_1^2+c_1^2) + B_2(b_1^2+c_1^2)\right]}{8(1+2\lambda)h_2^2}.$$

Since $B_1 > 0$, $h_2 > 0$, $0 \leq \lambda \leq 1$ and by using lemma 1, we obtain

$$|a_2|^2 \leq \frac{|\tau|(B_1 + |B_1 - B_2|)}{(1+2\lambda)h_2^2},$$

$$|a_2| \leq \frac{1}{h_2}\sqrt{\frac{|\tau|(B_1 + |B_1 - B_2|)}{(1+2\lambda)}}. \tag{33}$$

On the other by using (30) in (32) we obtain

$$\frac{2(1+2\lambda)a_2^2h_2^2}{\tau} = \left[\frac{1}{2}B_1(b_2+c_2) - \frac{2(B_1-B_2)(1+\lambda)^2a_2^2h_2^2}{\tau^2B_1^2}\right]. \tag{34}$$

Then

$$\frac{2a_2^2h_2^2}{\tau^2B_1^2}((B_1-B_2)(1+\lambda)^2 + \tau B_1^2(1+2\lambda)) = \frac{1}{2}B_1(b_2+c_2). \tag{35}$$

i.e.,

$$a_2^2 = \frac{\tau^2 B_1^3 (b_2 + c_2)}{4h_2^2((B_1 - B_2)(1+\lambda)^2 + \tau B_1^2(1+2\lambda))}.$$

By applying lemma 1, we obtain

$$|a_2|^2 \leq \frac{4\tau^2 B_1^3}{4h_2^2|(B_1 - B_2)(1+\lambda)^2 + \tau B_1^2(1+2\lambda)|},$$

$$|a_2| \leq \frac{|\tau| B_1 \sqrt{B_1}}{h_2\sqrt{|(B_1 - B_2)(1+\lambda)^2 + \tau B_1^2(1+2\lambda)|}}. \tag{36}$$

Now from (31), (34) and (36), we can find the bound for $|a_2|$.

Similarly, upon subtracting (28) from (26) and using (29) we get

$$\frac{2(1+2\lambda)a_3h_3}{\tau} - \frac{2(1+2\lambda)a_2^2h_2^2}{\tau} = \frac{1}{2}B_1(c_2 - b_2). \tag{37}$$

Now if we use (30) in (37) we obtain

$$\frac{2(1+2\lambda)a_3h_3}{\tau} = \frac{\tau B_1^2(1+2\lambda)(c_1^2 + b_1^2)}{4(1+\lambda)^2} + \frac{1}{2}B_1(c_2 - b_2). \tag{38}$$

$$a_3 = \frac{\tau^2 B_1^2(1+2\lambda)(c_1^2 + b_1^2)}{8(1+\lambda)^2(1+2\lambda)h_3} + \frac{\tau B_1(c_2 - b_2)}{4(1+2\lambda)h_3},$$

By applying lemma 1, we have

$$|a_3| \leq \frac{|\tau^2| B_1^2}{h_3(1+\lambda)^2} + \frac{|\tau| B_1}{h_3(1+2\lambda)}. \tag{39}$$

If we use (32) in (37), we get

$$\frac{2(1+2\lambda)a_3h_3}{\tau} = \left[\frac{1}{2}B_1(c_2 + c_2) - \frac{1}{4}B_1(b_1^2 + c_1^2) + \frac{1}{4}B_2(b_1^2 + c_1^2)\right], \tag{40}$$

$$a_3 = \frac{\tau\left(2B_1(c_2 + c_2) - B_1(b_1^2 + c_1^2) + B_2(b_1^2 + c_1^2)\right)}{8(1+2\lambda)h_3}.$$

Since $B_1 > 0$, $h_3 > 0$, $0 \leq \lambda \leq 1$ and by using lemma 1, we get

$$|a_3| \leq \frac{|\tau|(B_1 + |B_1 - B_2|)}{h_3(1+2\lambda)}. \tag{41}$$

Now from (39) and (41), we can find the bound for $|a_3|$.

□

## 4. Conclusions

If we take $\tau = 1$ and

$$\varphi = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \ (0 < \alpha \leq 1),$$

in Theorem 3, we conclude the following result which is an improvement of Theorem 1.

**Corollary 1.** *Let $f(z) \in \mathcal{B}_{\Sigma}(h, \alpha, \lambda)$ be of the form (1). Then*

$$|a_2| \leq \min\left[\frac{2\alpha}{h_2(1+\lambda)}, \frac{1}{h_2}\sqrt{\frac{4\alpha - 2\alpha^2}{(1+2\lambda)}}, \frac{2\alpha}{h_2\sqrt{(1+\lambda)^2 + \alpha(1+2\lambda-\lambda^2)}}\right]. \quad (42)$$

$$|a_3| \leq \min\left[\frac{4\alpha - 2\alpha^2}{h_3(1+2\lambda)}, \frac{4\alpha^2}{h_3(1+\lambda)^2} + \frac{2\alpha}{h_3(1+2\lambda)}\right]. \quad (43)$$

If we take $h(z) = \frac{z}{1-z}$ in Corollary 1 we obtain the following result which is an improvement of theorem obtained by Frasin and Aouf [11].

**Corollary 2.** *Let $f(z) \in \mathcal{B}_{\Sigma}(\alpha, \lambda)$ be of the form (1). Then*

$$|a_2| \leq \min\left[\frac{2\alpha}{(1+\lambda)}, \sqrt{\frac{4\alpha - 2\alpha^2}{(1+2\lambda)}}, \frac{2\alpha}{\sqrt{(1+\lambda)^2 + \alpha(1+2\lambda-\lambda^2)}}\right]. \quad (44)$$

$$|a_3| \leq \min\left[\frac{4\alpha - 2\alpha^2}{(1+2\lambda)}, \frac{4\alpha^2}{(1+\lambda)^2} + \frac{2\alpha}{(1+2\lambda)}\right]. \quad (45)$$

If we take $\lambda = 1$ in Corollary 2, then we have the following result which is an improvement of result obtained by Srivastava et al. [15].

**Corollary 3.** *Let $f(z) \in \mathcal{H}_{\Sigma}^{\alpha}$ be of the form (1). Then*

$$|a_2| \leq \min\left[\sqrt{\frac{4\alpha - 2\alpha^2}{3}}, \frac{2\alpha}{\sqrt{4+2\alpha}}\right]. \quad (46)$$

$$|a_3| \leq \min\left[\frac{4\alpha - 2\alpha^2}{3}, \alpha^2 + \frac{2\alpha}{3}\right]. \quad (47)$$

If we take $\tau = 1$ and

$$\varphi = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \ (0 \leq \beta < 1)$$

in Theorem 3, we conclude the following result which is an improvement of Theorem 2.

**Corollary 4.** *Let $f(z) \in \mathcal{B}_\Sigma(h, \beta, \lambda)$ be of the form (1). Then*

$$|a_2| \leq \min\left[\frac{2(1-\beta)}{h_2(1+\lambda)}, \frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(1+2\lambda)}}\right]. \tag{48}$$

$$|a_3| \leq \frac{2(1-\beta)}{h_3(1+2\lambda)}. \tag{49}$$

If we take $h(z) = \frac{z}{1-z}$ in Corollary 4 we obtain the following result which is an improvement of theorem obtained by Frasin and Aouf [11].

**Corollary 5.** *Let $f(z) \in \mathcal{B}_\Sigma(\beta, \lambda)$ be of the form (1). Then*

$$|a_2| \leq \min\left[\frac{2(1-\beta)}{(1+\lambda)}, \sqrt{\frac{2(1-\beta)}{(1+2\lambda)}}\right]. \tag{50}$$

$$|a_3| \leq \frac{2(1-\beta)}{(1+2\lambda)}. \tag{51}$$

If we take $\lambda = 1$ in Corollary 5 we obtain the following result which is an improvement of theorem obtained and studied by Srivastava et al. [15].

**Corollary 6.** *Let $f(z) \in \mathcal{H}_\Sigma(\beta)$ be of the*

$$|a_2| \leq \min\left[(1-\beta), \sqrt{\frac{2(1-\beta)}{3}}\right]. \tag{52}$$

$$|a_3| \leq \frac{2(1-\beta)}{3}. \tag{53}$$

## References


[1] Brannan, D. A., Clunie J. and Kirwan, W. E., Coefficient estimates for a class of starlike functions, *Can. J. Math.* 22 (1970) 476-485.

[2] Brannan, D. A. and Taha, T.S., On some classes of bi-univalent functions, KFAS Proceedings Series, vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, 53-60.

[3] Catas, A., Oros, G. I., Oros, G., Differential subordinations associated with multiplier transformations, Abstr. Appl. Anal. (2008), ID 845724:1-11.

[4] Chen, M., On the regular functions satisfying $\Re(f(z)/z) > \alpha$, *Bull. Inst. Math. Acad. Sinica* 3 (1975) 65-70.

[5] Chichra, P. N., New subclasses of the class of close-to-convex functions, *Proc. Am. Math. Soc.* (62) (1977) 37-43.

[6] Çağlar, M., Deniz, E. and Srivastava, H. M., Second Hankel determinant for certain subclasses of bi-univalent functions, *Turkish J. Math.* **41**, (2017) 694-706.

[7] Ding, S. S., Ling Y. and Bao, G. J., Some properties of a class of analytic functions, *J. Math. Anal. Appl.* 195 (1) (1995) 71-81.

[8] Duren, P. L., *Univalent Functions*, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[9] Dziok, J., Srivastava, H. M., Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.* 103 (1999) 1-13.

[10] El-Ashwah, R. M., Subclasses of bi-univalent functions defined by convolution, *J. Egypt. Math. Soc.* (2013)

[11] Frasin B. A. and Aouf, M. K., New subclasses of bi-univalent functions, *Appl. Math. Lett.* 24 (2011) 1569-1573.

[12] MacGregor, T. H., Functions whose derivative has a positive real part, *Trans. Am. Math. Soc.* 104 (1962) 532-537.

[13] Pommerenke, C., Univalent Functions, Vandenhoeck and Rupercht, Gottingen, 1975.

[14] Porwal S. and Darus, M., On a new subclass of bi-univalent functions, *J. Egyptian Math. Soc.* **21**, (2013) 190-193.

[15] Srivastava, H. M., Mishra A. K. and Gochhayat, P., Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* 23 (2010) 1188-1192.

[16] Srivastava, H. M., Bulut, S., Çağlar M. and Yağmur, N., Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat* **27** (5), (2013) 831-842.

[17] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some general subclasses of analytic and bi-univalent functions, *Afr. Mat.* **28**, (2017) 693-706.

[18] Srivastava, H. M., Gaboury S. and Ghanim, F., Initial coefficient estimates for some subclasses of $m$-fold symmetric bi-univalent functions, *Acta Math. Sci. Ser. B Engl. Ed.* **36**, (2016) 863-871.

[19] Srivastava, H. M., Joshi, B. S., Joshi S. and Pawar, H., Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, *Palest. J. Math.* **5**, (2016) Special Issue, 250-258.

[20] Srivastava, H. M., Sumer Eker S. and Rosihan Ali, M., Coefficient bounds for a certain class of analytic and bi-univalent functions, *Filomat* **29**, (2015) 1839-1845.

[21] Srivastava H. M. and Bansal, D., Coefficient estimates for a subclass of analytic and bi-univalent functions, *J. Egyptian Math. Soc.* **23**, (2015) 242-246.

[22] Srivastava, H. M., Gaboury S. and Ghanim, F., Coefficient estimates for some subclasses of $M$-fold symmetric bi-univalent functions, *Acta Univ. Apulensis Math. Inform.* **23**, (2015) 153-164.

[23] Srivastava, H. M., Sivasubramanian S. and Sivakumar, R., Initial coefficient bounds for a subclass of $m$-fold symmetric bi-univalent functions, *Tbilisi Math. J.* **7**, (2014) 1-10.

[24] Tang, Huo, Srivastava, H. M., Sivasubramanian S. and Gurusamy, P., The Fekete-Szego functional problems for some subclasses of $m$-fold symmetric bi-univalent functions, *J. Math. Inequal.* **10**, (2016) 1063-1092.

[25] Q.- H. Xu, Y.- C. Gui and H. M. Srivastava, Coefficient estimates for a Certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25**, (2012) 990-994.

[26] Xu, Q. -H., Xiao H. -G. and Srivastava, H. M., A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218** (23), (2012), 11461-11465.

[27] Zireh A. and Analouei Audegani, E., Coefficient estimates for a subclass of analytic and bi-univalent functions, *Bull. Iranian Math. Soc.* **42** (2016), 881-889.form (1). Then

*Current address*: Ahmad Motamednezhad (Corresponding author): Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran

*E-mail address*: a.motamedne@gmail.com

ORCID Address: http://orcid.org/0000-0001-6844-129X

*Current address*: Shahpour Nosrati: Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran

*E-mail address*: shahpournosrati@yahoo.com

ORCID Address: http://orcid.org/0000-0002-8127-4913

*Current address*: Sima Zaker: Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran

*E-mail address*: zaker.sima@yahoo.com

ORCID Address: http://orcid.org/0000-0002-3004-443X